BBC-003-1164003 Seat No. ________

M. Sc. (Sem. IV) Examination

July - 2021

Mathematics : CMT - 4003

(Number Theory - 2)

Faculty Code : 003

Subject Code : 1164003

Time : $2\frac{1}{2}$ Hours] [Total Marks : **70**

Instructions:

(1) Attempt any five questions from the followings.

(2) There are total ten questions.

(3) Each question carries equal marks.

1 Answer the following: [7 X 2 = 14] **14**

a) Express the rational numbers $\frac{1055}{8}$ and $\frac{554}{11}$ in continued fraction expansion.

b) Find the value of $< -2,4,4,3,3,1 >$ and $< 1,1,1,1,1,2 >$.

c) Find out, the values of $, r_3, r_4$ of $< 2,4,3,4,1,1 >$.

d) Find, four Primitive Pythagorean triplet (x, y, z) for which $z > 35$.

e) Express the numbers $\sqrt{2} + 1$and $\frac{\sqrt{5}-1}{2}$ in continued fraction expansion.

f) Define: a) Diophantine Equation and b) Simple Continued Fraction Expansion

g) If (x, y, z) is a Pythagorean Triplet then, show that, $gcd\ (y, z) = gcd(x, y, z)$

2) Answer the following: [7 X 2 = 14] **14**

a) Prove that, $h_n k_{n-1} - h_{n-1} k_n = (-1)^{n-1}$.

b) Define: i) Quadratic irrational and ii) Pell's equation with examples.

c) Write down the farey fractions [between 0 and 1] of 5^{th} and 6^{th} row.

d) Show that, there are infinitely many solutions (x, y) of $x^2 - dy^2 = 1$ in which k/y for $d > 1$ is not a perfect square and $k \geq 1$.

e) Find the number, whose expansion is $< 0,4,4,8,4,8,4,8, \ldots \ldots . . >$ and $< 2,2,2,2. \ldots . . >$.

f) Prove that, the g.c.d (x, y) where $\frac{x}{y}$ is a farey fraction of the n^{th} row is 1.

g) If $a > 1$ is a real and if $x + x^{-1} < \sqrt{5}$ then, show that,
$x < \frac{\sqrt{5}+1}{2}$ and $x^{-1} > \frac{\sqrt{5}-1}{2}$.

3 Answer the following: [2 X 7 = 14] **14**

1) State and prove, Hurwitz Inequality for continued fractions.
2) Prove that, if x is a rational number then its finite simple continued fraction expansion is always unique provided the last term is not equals to 1.

4 Answer the following: [2 X 7 = 14] **14**

1) Prove that, for any $n \geq 1$, there is a polynomial $f_n(x)$ with integer co-efficient of degree n and leading co-efficient 1 such that $f_n(2\cos\theta) = 2\cos n\theta$.
2) Prove that, the value of $f(x) = x^4 + x^3 + x^2 + x + 1$ is a perfect square for $x = -1, 0$ and 3 and for some values of $x, f(x)$ is not a perfect square.

5 Answer the following: [(5+4) + 5 = 14] **14**

1) i) Show that, if θ is a rational multiple of π then the only rational values $cos\theta$ can take are $0. \mp\frac{1}{2}. \mp 1$.
   ii) Check, whether the equation $x^2 - 18y^2 = 1$ and $x^2 - 18y^2 = -1$ has a solution or not. Justify your answer.
2) Prove that, $x^2 - 11y^2 = -1$ has no solution in integers.

6 Answer the following: [2 X 7 = 14] **14**

1) State and prove, the necessary and sufficient condition under which the continued fraction expansion of quadratic irrational is purely periodic.
2) Find, the first four positive solution of $x^2 - 7y^2 = 1$.

7 Answer the following: [2 X 7 = 14] **14**

1) Prove that, there are infinitely many positive integers n such that $\sum n$ is a perfect square.
2) Show that, the sequence $< a_0, a_1, a_2, \ldots, a_n >$ is a finite simple continued fraction if and only if its value is a rational number.

8 Answer the following: [2 X 7 = 14] **14**

1) Suppose (x_1, y_1) is a smallest positive solution of $x^2 - dy^2 = 1$ then prove that,

   i) $\left(x_1 + \sqrt{d}y_1\right)^n$ is a solution of it for $n \geq 1$.

   ii) Every positive solution is of the form $\left(x_1 + \sqrt{d}y_1\right)^n$ for some n.

2) If x is an irrational number and $\frac{a}{b}$ is a rational number with $(a, b) = 1$ and $b > 0$ such that, $\left|x - \frac{a}{b}\right| < \frac{1}{2b^2}$ then show that, $a = h_n$ and $b = k_n$; for some n.

9 Answer the following: [2 X 7 = 14] **14**

1) Prove that, if (x, y, z) is a primitive Pythagorean triplet then either

   i) x is odd and y is even or x is even and y is odd.

   ii) z is always odd.

2) Show that, the equation $x^4 = z^2 - y^4$ has no solution in integers.

10 Answer the following: [2 X 7 = 14] **14**

1) Suppose $< a_{0,} a_1 a_{2,} \ldots \ldots, a_n, \ldots .. >$ be an infinite sequence of integers with $a_i \geq 1$; for $i = 1,2,3, \ldots \ldots n$ then prove that, the subsequences r_{2j} and r_{2j-1} both converges to the same point, where h_j , k_j and r_j are defined as usual.

2) Suppose $C_n x^n + C_{n-1} x^{n-1} + \cdots + C_0$ is a polynomial with integer co-efficient and $\frac{s}{t}$ is a rational number with t is a positive and $(s, t) = 1$. If $\frac{s}{t}$ is a root of this polynomial then prove that, $s\ divide\ to\ C_0\ and$ $t\ divide\ to\ C_n$. Hence, deduce that, if a is an integer and $x^n = a$ has a rational root then it must be an integer.
